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Scaling theory of depinning in the Sneppen model

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We develop a scaling theory for the critical depinning behavior of the Sneppen interface model [Phys. Rev. Lett. **69**, 3539 (1992)]. This theory is based on a "gap" equation that describes the self-organization process to a critical state of the depinning transition. All of the critical exponents can be expressed in terms of two independent exponents, $\nu_{\parallel}(d)$ and $\nu_{\perp}(d)$, characterizing the divergence of the parallel and perpendicular correlation lengths as the interface approaches its dynamical attractor.

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The behavior of an interface that is driven in the presence of quenched random pinning forces appears in a wide variety of contexts. Familiar examples include fluid invasion in a porous medium [1], the motion of magnetic domain walls in the presence of quenched disorder [2], and various forms of growth phenomena where an interface between two media moves [3]. If this motion is extremely slow, the static and dynamic scaling behavior of the interface may be associated with a depinning transition. Here, we construct a scaling theory for the depinning transition by studying an interface model proposed by Sneppen [4]. We derive an equation for the Sneppen model that describes the self-organization process to a critical depinned state. All of the scaling relations discussed here follow as a consequence of this "gap" equation.

The depinning transition may be envisioned as follows: Below a critical driving force $F<F_c$ the interface is pinned in one of many possible metastable configurations by the random pinning forces. In response to a small increase in F, i.e., $F\rightarrow F+\Delta F$, the interface exhibits jerky motion in localized areas until eventually it gets pinned again at this new larger value of F. These avalanches of motion take the interface from one metastable state to another. They grow in size as F increases, until eventually for $F=F_c$ their size diverges, and the interface may move infinitely far from its original position.

The depinning of the interface at F_c can be described as a critical phenomenon with characteristic divergent length scales of the avalanches below the transition; while above it the interface moves with a finite velocity v, which vanishes as $v\sim(F-F_c)^{\beta}$. Hence, v corresponds to the order parameter for this transition. Critical behavior associated with depinning should be observed whenever an interface is driven with a vanishingly small velocity or becomes stuck. For example, if the force F decreases slowly from a value above F_c, eventually passing through F_c on its continuing decline, the interface becomes stuck at the point when $F=F_c$ and maintains this configuration forever. This situation may occur, for example, in an experiment where a long piece of paper, ignited along one edge, is burned in a container sealed with a finite amount of oxygen. The paper has fluctuations in its composition that correspond to the quenched random pinning forces. As the flame burns, the oxygen is depleted, and eventually the fire goes out. At this point, the interface separating the burned and unburned regions becomes stuck in a critical state of the depinning transition.

Recently, Sneppen [4] introduced a simple lattice model (model B of Ref. [4], referred to herein as the Sneppen model) that describes critical interface depinning. The unique feature of this model is that it self-organizes to the critical depinning state [5]; so that rather than having to fine-tune the driving force F, the Sneppen model finds F_c on its own. (This behavior was also found for a similar model constructed by Zaitsev [6] to describe low temperature creep phenomena such as dislocation glide.) In contrast to the flame experiment described above, however, the Sneppen model approaches F_c from the pinned state rather than from the moving state.

The Sneppen model, in 1+1 dimensions, is defined by an interface on a discrete lattice (x,h) where (a) a random pinning force $f(x,h)$ is assigned from a uniform probability distribution between zero and one. (b) Growth occurs by locating the site on the interface with the smallest random pinning force $f_{\min}$, and advancing the height at that site by one unit, $h\rightarrow h+1$. (c) After this advance has occurred, a constraint, $|(h_x-h_{x-1})|\leq 1$ for all x, is imposed, by advancing the heights of neighboring sites. Typically, a small number of sites are advanced during each *event*. Then, the site with the lowest random number is located once again, and the sequence of events continues indefinitely. It is straightforward to generalize this model to higher dimensions [7], where $\mathbf{x}$ is now a d-dimensional lattice coordinate.

Subsequently, Tang and Leschhorn [8,9] pointed out (i) that every allowed position of the interface in the (1+1)-dimensional Sneppen model corresponds to a path on a cluster of sites with values of f on it greater than or equal to $f_{\min}$ and (ii) that the Sneppen model in the critical state identifies exactly, from time to time, with a path on a directed percolation cluster at the critical point [10]. Thus the roughness of the Sneppen model in the critical state is $\chi=\nu_{\perp}/\nu_{\parallel}\simeq 0.633$, where $\nu_{\perp}\simeq 1.097$ and $\nu_{\parallel}\simeq 1.733$ are the correlation length exponents of directed percolation (DP) [11].

Generalizing the Tang-Leschhorn picture to d dimensions, for a given f_0 there exists a network consisting of all blocking surfaces that have $f(\mathbf{x},h)\geq f_0$ for all $\mathbf{x}$. This network represents all possible positions of the interface with $f_{\min}\geq f_0$. In 1+1 dimensions these surfaces are simply spanning paths on the infinite directed percolation cluster.

The minimal random pinning force that is chosen for each event, $f_{\min}(s)$, fluctuates in time, s. The f_0 avalanches are defined as the number of events separating instances when $f_{\min}(s)\geq f_0$. These avalanches correspond to filling in the voids in the f_0 network structure; i.e., going from one blocking surface, with $f(\mathbf{x},h)\geq f_0$ for all $\mathbf{x}$, to another. Thus, unlike the fractal DP cluster, the Sneppen avalanche is a compact object. Its volume, proportional to s, is given by $s\sim r_\perp r_\parallel^d$, where $r_\perp$ and $r_\parallel$ are parallel and perpendicular sizes in the network.

The mechanism of self-organization to the critical state in the Sneppen model is analogous to a mechanism recently developed [12] to describe the Bak-Sneppen evolution model [13]. Let us suppose that the Sneppen model is started in a flat configuration, $h(\mathbf{x},0)=0$ for all $\mathbf{x}$. We define the current gap $G(s)$ as the maximum of all the $f_{\min}(s')$ for $0\leq s'\leq s$. At the starting point $s=0$, the gap is very small, $O(L^{-d})$, where L is the linear size of the system. Then the gap gradually increases with time. By definition, $G(s)$ avalanches separate instances when the gap $G(s)$ jumps to its next higher value. The average size of the jumps in the gap is $[1-G(s)]/L^d$. Consequently, the growth of the gap versus time s is described by the following equation:

$$\frac{\partial G(s)}{\partial s}=\frac{1-G(s)}{L^d\langle s\rangle_{G(s)}}. \tag{1}$$

The average avalanche size $\langle s\rangle_{G(s)}$ is the average number of events to go from one $G(s)$ blocking surface to another. As the gap increases, the average avalanche size also increases, and eventually diverges as $G(s)\to f_c$; whereupon the Sneppen model is critical, and the distribution of $f(x,h)$ on the interface achieves stationarity. In the thermodynamic $L\to\infty$ limit, the density of sites with $f(x,h)<f_c$ vanishes, and the distribution of $f(x,h)$ is uniform above f_c.

The gap equation (1) defines the mechanism of approach to the self-organized critical (SOC) attractor for the Sneppen model. In order to solve it, we need to determine precisely how the average avalanche size $\langle s\rangle_{G(s)}$ diverges as the critical state is approached; i.e.,

$$\langle s\rangle_{G(s)}\sim[f_c-G(s)]^{-\gamma}. \tag{2}$$

This divergence is governed by the fixed point which describes scaling at and near criticality. In 1+1 dimensions the Tang-Leschhorn [8,9] mapping to DP shows that the correlation lengths near this fixed point are exactly those of DP. As a result, the critical exponent γ can be determined from DP (but does not correspond to the γ of DP). For the Sneppen model in $d>1$, the critical fixed point, or the value of γ, is not known at present. Nevertheless the gap equation explicitly shows that in any dimension $d\geq 1$, the Sneppen model exhibits SOC.

Using the following argument, the gap equation enables one to obtain all the critical exponents for the Sneppen model from two independent exponents, $\nu_\parallel$ and $\nu_\perp$. Equation (2) is valid when the parallel correlation length $\xi_\parallel$ is much less than L, so that the avalanche distribution has no system size corrections. Integrating Eq. (1) to times $s\gg L^d$ where Eq. (2) is valid gives

$$\Delta f=f_c-G(s)\sim\left(\frac{s}{L^d}\right)^{-1/(\gamma-1)}. \tag{3}$$

Since $\xi_\parallel\sim\Delta f^{-\nu_\parallel}$, where Δf is given in Eq. (3), the condition $\xi_\parallel<L$ holds as long as $s<s_{\text{transient}}\sim L^{\tilde d}$, where $\tilde d=d+(\gamma-1)/\nu_\parallel$. On the other hand, this transient time is the time that it takes the system to reach the first critical blocking surface which spans the system length. Geometrically this time is a volume between the initial interface $h(\mathbf{x})=0$ and the first blocking surface. So, $s_{\text{transient}}\sim L^{d+\nu_\perp/\nu_\parallel}$. For these two expressions for $s_{\text{transient}}$ to be consistent, one requires

$$\gamma=1+\nu_\perp. \tag{4}$$

Interestingly, this result for a different interface model was also derived by Narayan and Fisher [14]. Now, substituting Eq. (4) into (3), the approach to the SOC attractor is described by $\Delta f\sim(s/L^d)^{-1/\nu_\perp}$.

As was pointed out in [9], the distribution of avalanche sizes scales as

$$P(s)\sim s^{-\tau}\mathcal{F}(s/\Delta f^{-\nu}), \tag{5}$$

near the critical point. Given the relation $s\sim r_\parallel^d r_\perp$, the exponent ν can be expressed as

$$\nu=d\nu_\parallel+\nu_\perp. \tag{6}$$

Since $\gamma=\nu(2-\tau)$, the exponent τ is given by

$$\tau=1+\frac{d-\frac{1}{\nu_\parallel}}{d+\nu_\perp/\nu_\parallel}. \tag{7}$$

The exponent γ determines another important steady-state exponent of the model: the probability $P(f_{\min}=f_0)$ that the chosen minimal random number is equal to f_0. From the definition of the f_0 avalanche, the probability to observe $f_{\min}\geq f_0$ is equal to $1/\langle s\rangle_{f_0}$. This immediately gives [15]

$$P(f_{\min}=f_0)\sim(f_c-f_0)^{\gamma-1}=(f_c-f_0)^{\nu_\perp}. \tag{8}$$

The above relation enables us to correct the exponent γ_{TL} used by Leschhorn-Tang [9] to describe the distribution of distances between subsequent active sites. From Eq. (10) in Ref. [9] one finds the relation $\gamma_{\text{TL}}=1+\gamma/\nu_\parallel=1+(1+\nu_\perp)/\nu_\parallel$, instead of $\gamma_{\text{TL}}=1+2/\nu_\parallel$. In 1+1 dimensions, the Leschhorn-Tang prediction $\gamma_{\text{TL}}\simeq 2.16$ is just within the margin of error of their measured value $\gamma_{\text{TL}}=2.20\pm0.05$ and disagrees with the value of 2.25 ± 0.05 measured in Ref. [16]. However, using the values from DP in our relation gives $\gamma_{\text{TL}}\simeq 2.21$, which appears to agree with both numerical results. Furthermore, we suggest

that the relation $\gamma_{TL}=1+\gamma/\nu_{\parallel}$ holds in any dimension. More numerical tests are needed, though, in order to completely clarify this situation.

In addition, our scaling relations [Eqs. (4), (6), and (7)], as well as the approach to the SOC attractor, disagree with a previous scaling theory for the 1+1 Sneppen model by Olami, Procaccia, and Zeitak [15]. Olami, Procaccia, and Zeitak introduce an additional scaling relation, from which they find $\nu_{\parallel}\simeq 1.58$, and $\nu_{\perp}=1$. This is inconsistent with the very careful numerical studies of Leschhorn and Tang [9], supporting their original proposal that the correlation length exponents are those of 1+1 DP. The data [9] for the avalanche size distribution exhibit excellent data collapse after rescaling according to (5). The exponents that were used for this data collapse are $\nu=\nu_{\parallel}+\nu_{\perp}=2.83$, which is exactly the same as our Eq. (6), and a numerically determined value $\tau=1.25$, compared with our prediction $\tau\simeq 1.26$. These values are quite far from the Olami, Procaccia, and Zeitak predictions $\nu\simeq 2.58$, $\tau\simeq 1.22$.

In 2+1 dimensions, our exponent relation for τ agrees with Ref. [17] and is consistent with the numerical simulations in 2+1 dimensions [7]. Their numerical results, $\tau=1.45\pm 0.03$ and $\nu_{\perp}/\nu_{\parallel}=0.5\pm 0.03$, and our scaling relations (5) and (8) give $\gamma-1=\nu_{\perp}=0.57\pm 0.05$, which is not far from the measured value of 0.5 ± 0.05.

In order to make a comparison with actual physical systems, it is important to note that an avalanche can be considered as a branching process where particles correspond to activity in the system. For a given $G(s)$ avalanche, particles are all sites which have pinning forces less than $G(s)$, i.e., those that will move before the avalanche dies out. The Sneppen branching process consists of picking the particle with the lowest random pinning force. However, it is clear that within a given branching process the scaling behavior is not altered by changing the order in which the particles are chosen, since the interface is constrained ultimately to pass from one blocking surface to another. (These blocking surfaces are manifestly independent of the order in which the unstable sites are chosen.) As a result, all particles in a branching process may be updated in parallel, rather than choosing the one with the smallest random pinning force. This latter situation corresponds to the physical picture described in the introduction. A model with such parallel rules of dynamics was studied in [18]. Aside from the time redefinition, the Sneppen mechanism to self-tune $F\rightarrow F_c$ *does not alter* the critical behavior from the case where an external F is quasistatically increased from $F=0$ to F_c, and the active sites are updated simultaneously. In this latter case, the gap $G(s)$ in Eq. (1) corresponds to the current value of F.

Returning to the flame experiment described earlier, we note that when the fire front is moving rapidly, its behavior above a certain length scale is believed to be determined by the Kardar-Parisi-Zhang [19] equation, so that its roughness $\chi=\frac{1}{2}$. However, this length scale diverges as F approaches the depinning threshold F_c from above. In fact, it was shown [18] that the critical index of this diverging length is the same $\nu_{\parallel}\simeq 1.733$ on both sides of the transition.

It is plausible that the Sneppen model describes a wide universality class of depinning phenomena [4,9] including the depinning of the flame front. Consequently, we conjecture that when the fire goes out, the burned edge that remains has the roughness of the Sneppen model, $\chi\approx 0.63$. This frozen front is much rougher than its fast-moving precursors. In this sense, the Sneppen model, as well as the extinguishing flame front, can be viewed as self-organized criticality [5]: in both cases the process leads to an interface of "anomalous" roughness, characteristic of only the critical point for depinning. In fact, experiments on "flameless" fire fronts in paper [20], prepared so that the flame front moves rather slowly, only a few millimeters per second, have found anomalous roughness, $\chi=0.7\pm 0.03$. We suspect that, below a certain length scale, such slow-moving fronts in a random medium are described by the Sneppen model.

In summary, we have shown that the Sneppen interface model exhibits self-organized criticality in any dimension $d\geq 1$. Our scaling theory is based on the behavior of the gap equation as the system approaches the steady state. Both static and dynamic exponents near the depinning transition can be determined from two independent exponents $\nu_{\parallel}(d)$ and $\nu_{\perp}(d)$. In one dimension, these correlation length exponents are given by directed percolation. Driving the Sneppen model with parallel dynamics does not alter the critical behavior, but requires the force to be tuned.

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